

Insertion and Sorting in a Sequence of Numbers Minimizing the Maximum Sum of a Contiguous Subsequence*

Ricardo C. Corrêa, Pablo M. S. Farias[†]

ParGO Research Group[‡]
Universidade Federal do Ceará
Campus do Pici, Bloco 910
60440-554 Fortaleza, CE, Brazil
{correa,pmsf}@lia.ufc.br

Críston P. de Souza[§]

ParGO Research Group[‡]
Universidade Federal do Ceará
Campus de Quixadá
63900-000 Quixadá, CE, Brazil
cristonsouza@lia.ufc.br

Abstract

Let A be a sequence of $n \geq 0$ real numbers. A subsequence of A is a sequence of contiguous elements of A . A *maximum scoring subsequence* of A is a subsequence with largest sum of its elements, which can be found in $O(n)$ time by Kadane's dynamic programming algorithm. We consider in this paper two problems involving maximal scoring subsequences of a sequence. Both of these problems arise in the context of buffer memory minimization in computer networks. The first one, which is called INSERTION IN A SEQUENCE WITH SCORES (ISS), consists in inserting a given real number x in A in such a way to minimize the sum of a maximum scoring subsequence of the resulting sequence, which can be easily done in $O(n^2)$ time by successively applying Kadane's algorithm to compute the maximum scoring subsequence of the resulting sequence corresponding to each possible insertion position for x . We show in this paper that the ISS problem can be solved in linear time and space with a more specialized algorithm. The second problem we consider in this paper is the SORTING A SEQUENCE BY SCORES (SSS) one, stated as follows: find a permutation A' of A that minimizes the sum of a maximum scoring subsequence. We show that the SSS problem is strongly NP-Hard and give a 2-approximation algorithm for it.

1 Introduction

Let the elements of a sequence A of $n \geq 0$ real numbers be denoted by a_1, a_2, \dots, a_n . Then, A is the sequence $\langle a_1, a_2, \dots, a_n \rangle$ (which is $\langle \rangle$ if $n = 0$) and its size is $|A| = n$. A subsequence of A defined by indices $0 \leq i \leq j \leq n$ is denoted by A_i^j , which equals either $\langle \rangle$, if $i = j$, or the sequence $\langle a_{i+1}, \dots, a_j \rangle$ of contiguous elements of A , otherwise (see Figure 1 for an example). Let $score(A_i^j) = \sum_{k=i+1}^j a_k$ stand for the sum of elements of A_i^j (we consider $score(\langle \rangle) = 0$). A *maximum scoring subsequence* of A is a subsequence with largest score. The MAXIMUM SCORING SUBSEQUENCE (MSS) problem is that of finding a maximum scoring subsequence of a given sequence A . The MSS problem can be solved in $O(n)$ time by Kadane's dynamic programming algorithm, whose essence is to consider A as a concatenation $\langle A_0^{j_1}, A_{i_2=j_1}^{j_2}, \dots, A_{i_\ell}^{j_\ell} \rangle$ of appropriate subsequences, called *intervals*, and to determine S_k as a maximum scoring subsequence of $A_{i_k}^{j_k}$, for all $k \in \{1, 2, \dots, \ell\}$. Defining each interval $A_{i_k}^{j_k}$ – with the possible exception of the last one – to be such that $score(A_{i_k}^{j_k}) < 0$ and $score(A_{i_k}^{j'}) \geq 0$, for all $i_k \leq j' < j_k$, then the largest score subsequence among $\{S_1, S_2, \dots, S_\ell\}$ is a maximum scoring subsequence of A [1, 2]. The *value* of A is $score^*(A) = score(S)$, for any maximum scoring subsequence S of A .

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[‡]<http://www.lia.ufc.br/~pargo>

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$$A = \begin{array}{c} 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ 16 \ 17 \ 18 \ 19 \ 20 \ 21 \ 22 \ 23 \ 24 \ 25 \ 26 \ 27 \ 28 \ 29 \\ \hline 1 \ 2 \ -3 \ 3 \ -1 \ -4 \ 3 \ -4 \ \boxed{4 \ 6 \ -5 \ -5 \ -5 \ 2 \ 4 \ -2 \ 5} \ 3 \ 0 \ -6 \ -4 \ 3 \ 2 \ -4 \ -6 \ 9 \ 2 \ -3 \ -2 \end{array}$$

$$A_8^{17}, \text{score}(A_8^{17}) = 4$$

Figure 1: An example of a sequence and a subsequence. A maximum scoring subsequence is A_{13}^{18} and $\text{score}^*(A) = 12$.

The MSS problem has several applications in practice, where maximum scoring subsequences correspond to various structures of interest. For instance, in Computational Biology, in the context of certain amino acid scoring schemes and several other applications mentioned in [3, 4]. In such a context, it may also be useful to find not only one but a maximal set of non-overlapping maximum scoring subsequences of a given sequence A . This can be formalized as the ALL MAXIMAL SCORING SUBSEQUENCES problem, for which have been devised a linear sequential algorithm [4], a PRAM EREW work-optimal algorithm that runs in $O(\log n)$ time and makes $O(n)$ operations [5] and a BSP/CGM parallel algorithm which uses p processors and takes $O(|A|/p)$ time and space per processor [6]. The MSS problem has also been generalized in the direction of finding a list of k (possibly overlapping) maximum scoring subsequences of a given sequence A . This is known as the k MAXIMUM SUMS [7] and for a generalization of it an optimal $O(n+k)$ time and $O(k)$ space algorithm has been devised [8, 9]. An optimal $O(n \cdot \max\{1, \log(k/n)\})$ algorithm has also been developed for the related problem of selecting the subsequence with the k -th largest score [9].

Sequences of numbers can also model buffer memory usage in a node of a computer network. In this case, the absolute value of a number models the local memory space required to store a corresponding message after its reception and before it is resent through the network (in practice, there are additional cases in which the message is produced or consumed locally; these situations are ignored in this high level description for the sake of simplicity of exposition). This behavior can be described more generally as the execution of tasks (sending or receiving messages), each of which is associated with a (positive or negative) cost that corresponds to the additional units of resources (local memory space) that are occupied after its execution. Receiving a message results in a positive cost, while sending a message can be viewed as effecting a negative cost. In this context, finding maximum scoring subsequences of sequences defining communications between the nodes of a network corresponds to finding the greatest buffer usage in each node [10]. Moreover, when the intention is to find an ordering for these communications with the aim of minimizing the resulting memory usage, then we are left with the problem of sorting the communications so as to minimize the maximum renewal cumulative cost.

We consider in this paper two problems related to the MSS. The first one, which is called INSERTION IN A SEQUENCE WITH SCORES (ISS), consists in inserting a given real number x in A in such a way to minimize the maximum score of a subsequence of the resulting sequence. The operation of *inserting* x in A is associated with an *insertion index* $p \in \{0, \dots, n\}$ and the *resulting sequence* $A^{(p)} = \langle A_0^p, x, A_p^n \rangle$, that is, the sequence obtained by the concatenation of A_0^p , x , and A_p^n . The objective of the ISS problem is to determine an insertion index p^* that minimizes $\text{score}^*(A^{(p^*)})$, which can be easily done in $O(n^2)$ time and $O(n)$ space by successively using Kadane's algorithm to compute the maximum scoring subsequence of $A^{(0)}, \dots, A^{(n)}$. We show in this paper that we can do better. More precisely, we show that the ISS problem can be solved in linear time.

The ISS problem can be approached more specifically depending on the value of x . The case $x = 0$ is trivial since $\text{score}^*(A^{(p)}) = \text{score}^*(A)$ independently of the value of p , which means that any insertion index p is optimal for A . If $x < 0$, then $\text{score}(A^{(p)}) < \text{score}(A)$, for all insertion indices $p \in \{0, 1, \dots, n\}$. Intuitively, then, x has to be inserted inside some maximum scoring subsequence $S = A_i^j$ of A , in an attempt to reduce the value of $A^{(p)}$ with respect to that of A . Even though the value of $A^{(p)}$ cannot be smaller than $\text{score}^*(A)$ in certain cases (for instance, if S has only one positive element, or $\text{score}^*(A_0^i) = \text{score}(S)$, or $\text{score}^*(A_j^n) = \text{score}(S)$, then all insertion indices are equally good for A since $\text{score}^*(A^{(p)}) = \text{score}^*(A)$ for any particular choice of p), we describe an $O(n)$ time and space algorithm to determine a best insertion position in a maximum scoring subsequence of A , provided that x is negative.

Showing that the ISS problem can be solved in linear time is a more complex task when $x > 0$. Inserting x inside a maximum scoring subsequence S of A will certainly lead to a subsequence S' of $A^{(p)}$ such that $\text{score}(S') > \text{score}(S)$ (this may happen even if x is inserted outside S). Intuitively, therefore, we should choose an insertion position where x can only “contribute” to subsequences whose scores are as small as possible. Computing the necessary information for this in $O(n)$ time may seem involving at first, but we can make things simpler by considering the partition into intervals of A (the same used in Kadane’s algorithm). The idea is to determine the interval $A_{i_k}^{j_k}$ having an optimal insertion index. The difficulty to accomplish this task in linear time stems from the fact that computing $\text{score}^*(A^{(p)})$ when p is an insertion index in an interval $A_{i_k}^{j_k}$ may involve one or more intervals other than $A_{i_k}^{j_k}$. We overcome this difficulty by means of a dynamic programming approach.

The second problem we consider in this paper is the SORTING A SEQUENCE BY SCORES (SSS), stated as follows: given the sequence A , find a permutation A' of A that minimizes $\text{score}^*(A')$. The SSS problem is the particular case of the SEQUENCING TO MINIMIZE THE MAXIMUM RENEWAL CUMULATIVE COST problem for which the partial order is empty [11]. It is mentioned in [11] that the SSS problem has been proved to be strongly NP-hard by means of a transformation from the 3-PARTITION problem. Indeed, a straightforward reduction from 3-PARTITION yields that the SSS problem remains NP-hard in the strong sense even if all negative elements in A are equal to $-s$ and every positive element a_i is such that $s/4 < a_i < s/2$, for some number s (more details are given in Section 5). On the other hand, it is known that the SSS problem becomes polynomially solvable if all positive elements are equal to some number s' independent from s [11].

Usually, sorting problems are closely related to insertion ones in the sense that one could expect that an appropriate sequence of insertions would produce a good sorting. In this sense, ISS and SSS problems can be related as follows. Let MISS be the problem of, given sequences $A = \langle a_1, \dots, a_n \rangle$ and $X = \langle x_1, \dots, x_k \rangle$, $k \geq 1$, finding a sequence A' which results from an insertion of the elements of X into A and which minimizes $\text{score}^*(A')$. Note that, since finding an optimal insertion of the elements of a sequence X into an empty sequence A implies finding an optimal permutation of X , then the MISS problem is NP-hard. Nevertheless, the following recursive algorithm for the MISS problem, which we call MULTIINSERT, can be turned into an exact algorithm for the SSS problem: if $k = 0$, then return A ; if $k = 1$, then return the sequence resulting from the insertion of x_1 in A ; otherwise, compute the sequence B_i , for each $i \in \{0, \dots, n\}$, which results from the insertion of x_1 in position i of A , as well as sequence $C_i = \text{MULTIINSERT}(B_i, \langle x_2, \dots, x_k \rangle)$, and then return C_i for some i which minimizes $\text{score}^*(C_i)$. A consequence of our linear algorithm for the ISS problem is then that MULTIINSERT can be made to run in $O(f(n, k))$ time, where $f(n, 0) = 1$, $f(n, 1) = n$ and $f(n, k) = (n+1) \cdot f(n+1, k-1) = O(((n+k)! - (n+k-1)!)/n!)$ if $k \geq 2$. Whether or not there are faster algorithms for the MISS problem is a subject for further investigations.

We present in this paper a $(1 + M/\text{score}^*(A))$ -approximation algorithm for the SSS problem, where M is the maximum element in A , which runs in $O(n \log n)$ time. For the general case of the SSS problem, since $\text{score}^*(A) \geq M$, this algorithm has approximation factor of 2, and we show that this factor is tight. However, for a more particular case, still strongly NP-hard, where the elements of A are bounded, from below and above, by linear functions of $\text{score}(A)$, the approximation factor of this same algorithm becomes $3(n+1)/2n$, for $n \geq 3$.

We organize the remaining of the text as follows. Section 2 states some useful properties of maximum score subsequences for later use. In Section 3 and Section 4 we then present our solutions to the ISS problem for the cases where the inserted number x is negative and positive, respectively. Section 5 contains our results on the SSS problem, and Section 6 finally provides conclusions and directions for further investigations.

2 Preliminaries on the ISS problem

Let us establish some simple and useful properties of sequence A and a subsequence A_i^j , for $0 \leq i \leq j \leq n$. We start with three properties that give a view of minimal (with respect to inclusion) maximum scoring

subsequences. Let a *prefix* (*suffix*) of A_i^j be a subsequence $A_{i'}^{j'}$ ($A_{i'}^{j'}$), with $i \leq j' \leq j$ ($i \leq i' \leq j$).

Fact 1. If A_i^j is a maximum scoring subsequence of A , then its prefixes and suffixes have all nonnegative scores, otherwise a larger scoring subsequence can be obtained by deleting a prefix or a suffix of negative score. Conversely, $\text{score}(X) \leq 0$, where X is any suffix of A_0^i or prefix of A_j^n , otherwise a larger scoring subsequence can be obtained by concatenating A_i^j with a suffix of A_0^i or prefix of A_j^n of positive score.

Fact 2. If A_i^j is a maximum scoring subsequence of A , then there is a maximum scoring subsequence of A_i^j which is a prefix (suffix) of A_i^j .

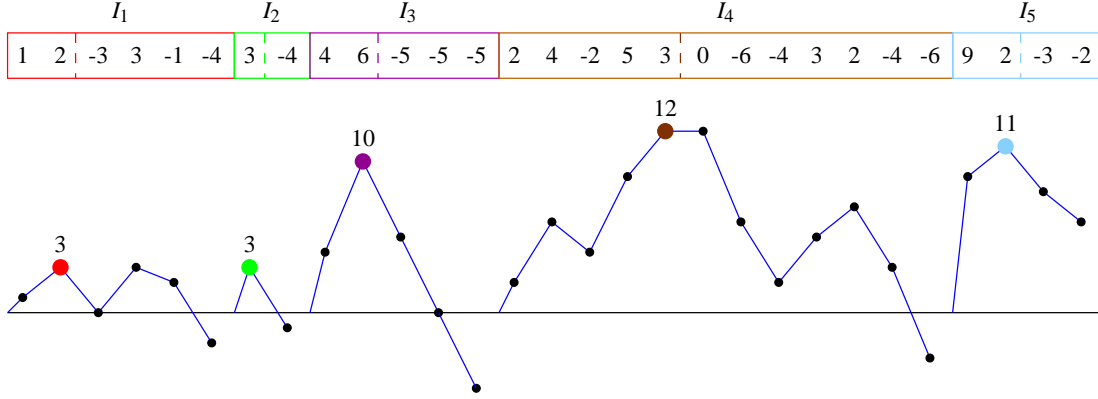


Figure 2: Partition into intervals of the sequence in Figure 1. For each interval, the score of its prefixes is indicated, as well as its maximum scoring subsequence.

The definitions in the sequel are illustrated in Figure 2. The subsequence A_i^j is an *interval* if $\text{score}(A_i^j) < 0$ or $j = n$, and $\text{score}(A_i^j) \geq 0$, for all $i \leq j' < j$. The *partition into intervals* of A is the concatenation $\langle I_1 = A_0^{j_1}, I_2 = A_{j_1}^{j_2}, \dots, I_\ell = A_{j_{\ell-1}}^{j_\ell} \rangle$ of the ℓ maximal intervals of A . Such a partition is explored in Kadane's algorithm due to the fact that a maximum scoring subsequence of A is a subsequence of some of its intervals.

Fact 3. If A_i^j is a maximum scoring subsequence of interval I_k and $A_{i'}^{j'}$ is a prefix (suffix) of A_i^j such that $\text{score}(A_{i'}^{j'}) = 0$, then $A_i^j \setminus A_{i'}^{j'}$ is a maximum scoring subsequence of I_k .

While the previous properties are general for every sequence, the next one is more specific to the resulting sequence of an insertion. Recall that x stands for the real number given as input to the ISS problem. Assume that the insertion index p is such that $i_k \leq p < j_k$, which means that x is inserted in I_k .

Fact 4. The score of all elements of I_k whose indices are greater than p are affected by the insertion of x in the following way: for every $p < q \leq j_k + 1$, $\text{score}(A_{i_k}^{(p)q}) = \text{score}(A_{i_k}^{q-1}) + x$.

This fact is the reason why the discussion of cases $x < 0$ and $x > 0$ is carried out separately in the two next sections. For the positive case, since all prefixes of I_k have nonnegative scores (Fact 1), consecutive intervals may be merged in the resulting sequence, provided that x is large enough to make $\text{score}(A_{i_k}^{(p)j_k+1}) > 0$. For instance, consider interval I_1 in Figure 2. The insertion of $x = 6$ at the very end of this interval (i.e., at insertion position $p = j_1 - 1 = 5$) creates the subsequence $\langle A_{i_1=0}^5, 6, -4 \rangle$ and the new interval $\langle A_0^5, 6, -4, I_2, I_3 \rangle$. On the other hand, for the negative case, the insertion of x may split I_k into two or more intervals if there exists $p \leq q \leq j_k$ such that $\text{score}(A_{i_k}^{(p)q}) < 0$, in which case $A_{i_k}^{(p)q}$ is an interval of $A^{(p)}$ but $A_{i_k}^{(p)j_k}$ is not. Again in Figure 2, the insertion of $x = -6$ between the elements -2 and 5 of interval I_4 splits it into 3 intervals, namely $\langle 2, 4, -2, -6 \rangle$, $\langle 5, 3, 0, -6, -4 \rangle$, and $\langle 3, 2, -4, -6 \rangle$.

3 Inserting $x < 0$

As already mentioned in the Introduction, solving the ISS problem when $x < 0$ corresponds to insert x in some maximum scoring subsequence A_i^j . According to Fact 3, we assume that A_i^j is minimal with respect to inclusion. What remains to be specified is the way to find an appropriate insertion index in A_i^j . The algorithm in the sequel is based on the fact that inserting x inside A_i^j divides the latter in its left (a prefix of A_i^j) and right (a suffix of A_i^j) parts, and different choices of p may lead to different values of $A^{(p)}$, as depicted in Figure 3. One solution is then to simply try out each possibility for p in the range from $i + 1$ to $j - 1$, computing the maximum between the values of the resulting left and right parts of A_i^j . Doing this straightforwardly takes $\Theta(n^2)$ time, since we would run Kadane's algorithm twice for each possible value of p (once for the left and once more for the right part of A_i^j). Fortunately, this can also be easily done in $O(n)$ time, since a left-to-right traversal of A_i^j can be used to compute (and store) the values of all possible left parts of A_i^j , and a further right-to-left traversal can be used to compute the values of all possible right parts of A_i^j . This strategy is materialized in Algorithm 1, which employs a version of Kadane's algorithm as a sub-routine returning the indices i and j and the score of the minimal maximum scoring subsequence considered.

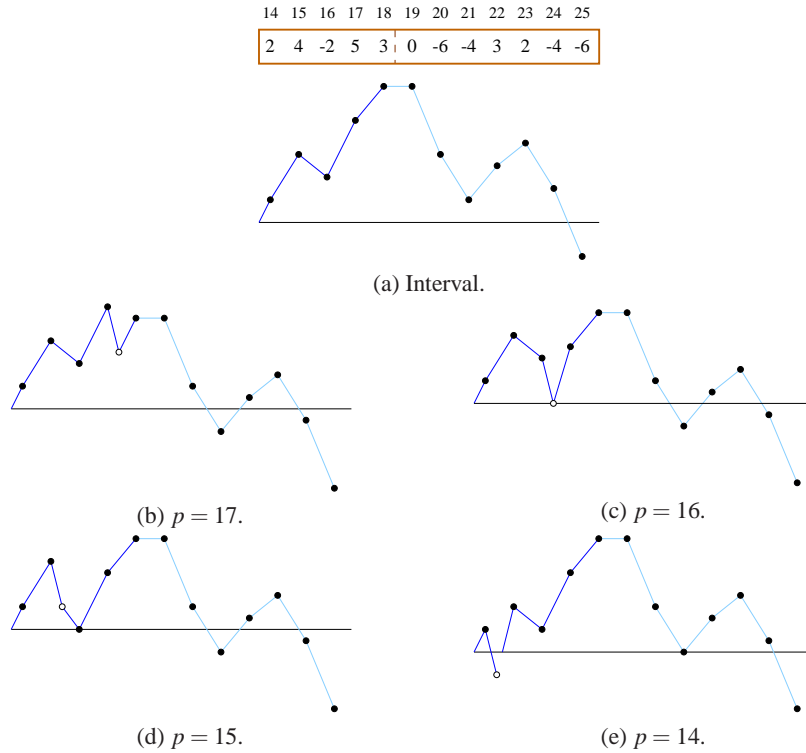


Figure 3: Possible insertion positions in the interval I_4 of the example in Figure 2 for $x = -4$.

Three main variables are used in the algorithm, with the following interpretation. Variable SX plays different roles in loops at lines 6 and 12. In the first case, it stores the prefix sums of A_i^j , while in the second, suffix sums. Array L and variable R store the maximum scores of prefixes and suffixes of A_i^j , respectively.

Lemma 1. *Algorithm 1 is correct, i.e. $\text{INSERTIONOFNEGATIVE}(A, x)$ returns an optimal insertion index p , provided that $x < 0$. In addition, it runs in $O(n)$ time and space.*

Proof. The trivial cases $n = 0$, $j \leq i + 1$, and $\text{score}(A_i^j) = 0$ are properly handled at line 3. Then, assume that $n > 0$, $j > i + 1$, and $\text{score}(A_i^j) > 0$.

Let $p \in \{i + 1, \dots, j - 1\}$ be the value returned by the algorithm and $p' \neq p$ be another arbitrary insertion index. We show that $\text{score}^*(A^{(p)}) \leq \text{score}^*(A^{(p')})$. Let in addition T be a maximum scoring

Algorithm 1: INSERTIONOFNEGATIVE

Input: an array A of $n \geq 0$ real numbers and a real number $x < 0$

Output: an optimal insertion index for A

```
1  $(i, j, s) \leftarrow \text{KADANE}(A)$ 
2
3 if  $j \leq i + 1$  or  $s = 0$  then return  $i$ 
4  $SX \leftarrow a_{i+1}$ 
5  $L[1] \leftarrow SX$ 
6 for  $k \leftarrow 2, \dots, j - i$  do                                // Largest scores of prefixes of  $A_i^j$ 
7    $SX \leftarrow SX + a_{i+k}$ 
8    $L[k] \leftarrow \max\{L[k-1], SX\}$ 
9  $m \leftarrow L[j-i]$ 
10  $SX \leftarrow 0$ 
11  $R \leftarrow 0$ 
12 for  $k \leftarrow j - i, \dots, 2$  do                                // Largest scores of suffixes of  $A_i^j$ 
13    $SX \leftarrow SX + a_{i+k}$ 
14    $R \leftarrow \max\{R, SX\}$ 
15   if  $\max\{L[k-1], R\} < m$  then
16      $m \leftarrow \max\{L[k-1], R\}$ 
17      $p \leftarrow i + k - 1$ 
18 return  $p$ 
```

subsequence of $A^{(p)}$, minimal with respect to inclusion. Note that $T \neq \langle \rangle$ since $\text{score}^*(A) = \text{score}(A_i^j) > 0$. Moreover, by Fact 1, x is neither the first nor the last element of T . So, let y and z be such that $T_0^1 = \langle a_{y+1} \rangle$ and $T_{|T|-1}^{|T|} = \langle a_z \rangle$. The first case to be analyzed is when x is in T , i.e. $y < p < z$ (Figure 4(a)). In this case, by Fact 1 and the minimality of A_i^j and T , $y = i$ and $z = j$ or, in other words, $T = \langle A_i^p, x, A_p^j \rangle$. The elements of A_i^j also form, perhaps with the occurrence of x at some position, a subsequence T' of $A^{(p')}$, and since $x < 0$, we conclude that $\text{score}(T') \geq \text{score}(T)$ (equality holds if $y < p' < z$). Then $\text{score}^*(A^{(p)}) = \text{score}(T) \leq \text{score}(T') \leq \text{score}^*(A^{(p')})$, as claimed.

Assume that $p \notin \{y, \dots, z\}$. If T 's elements also form a subsequence of $A^{(p')}$ (more precisely, $p' \notin \{y+1, \dots, z-1\}$), then $\text{score}^*(A^{(p)}) = \text{score}(T) \leq \text{score}^*(A^{(p')})$, as desired. Then, assume that $p' \in \{y+1, \dots, z-1\}$. If A_i^j and T are disjoint, then A_i^j is also a subsequence of $A^{(p')}$. It turns out that $\text{score}^*(A^{(p')}) \leq \text{score}^*(A) = \text{score}(A_i^j)$ yields $\text{score}^*(A^{(p')}) = \text{score}(A_i^j) = \text{score}^*(A) \geq \text{score}^*(A^{(p)})$.

Finally, we are left with the case when A_i^j and T are not disjoint (Figure 4(b)), which requires a more detailed analysis of Algorithm 1. By Fact 1 and the minimality of T , either $y = i$ or $z = j$. Without loss of generality, let us suppose the first equality, since the other one is analogous. The values computed in the loops of lines 8 and 14 for each $L[k-1]$ and R correspond to $\text{score}^*(A_{i+k-1}^j)$ and $\text{score}^*(A_{i+k-1}^j)$, respectively, by the property established in Fact 2. Due to line 15, we have that $\max\{\text{score}^*(A_i^{p'}), \text{score}^*(A_{p'}^j)\} \geq \text{score}^*(A_i^p) = \text{score}(T)$. The result follows since both $A_i^{p'}$ and $A_{p'}^j$ are subsequences of $A^{(p')}$.

The complexities stem directly from the facts that the algorithm employs one array of size $O(n)$ and performs two disjoint $O(n)$ -time loops. \square

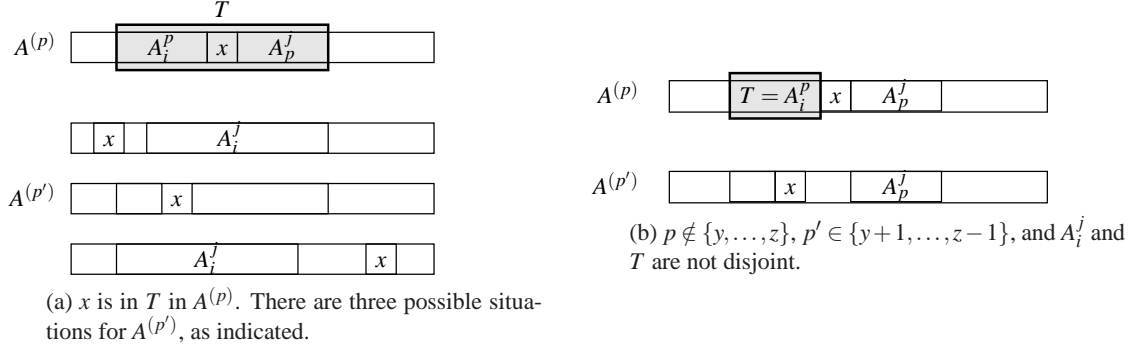


Figure 4: Cases of proof of Lemma 1.

4 Inserting $x > 0$

The discussion in this section is based on the partition into intervals $\langle I_1, I_2, \dots, I_\ell \rangle$ of A . For the sake of convenience, we assume that $a_n = 0$ (observe that this can be done without loss of generality since appending a new null element to A does not alter the scores of the suffixes of A), which means that $\text{score}(I_\ell) \geq 0$. A particularity of this positive case, which is derived from Fact 4, is the following: for every interval I_k , index $j_k - 1$ is at least as good as any other insertion index in this interval. Thus, an optimal insertion index exists among $j_1 - 1, j_2 - 1, \dots, j_\ell - 1$, corresponding each one of these indices to one interval of the partition into intervals of A . If $p = j_k - 1$ is chosen as the insertion index, then the resulting interval in $A^{(p)}$ (which may correspond to a merge of several contiguous intervals of A in the sense of Fact 4) is referred as an *extended interval, relative to I_k* and denoted by $I^{(k)}$. If $I_{k'}$ is one of the intervals which are merged to produce $I^{(k)}$, then $I_{k'}$ is a *subinterval* of $I^{(k)}$. In the remaining of this section, we show a linear time algorithm to compute $\text{score}^*(I^{(k)})$, for all $k \in \{1, 2, \dots, \ell\}$. Clearly, the smallest of these values is associated with the optimal insertion index for x .

For each k , computing $\text{score}^*(I^{(k)})$ by means of Kadane's algorithm takes $\Theta(n)$ time. Therefore, the exhaustive search takes quadratic time in the worst case. However, as depicted in Figure 5, by graphically aligning the scores of the prefixes of the extended intervals with respect to the intervals of A , one can visualize some useful observations in connection with these curves which are explored in the algorithm described in the sequel. Let the sequence of negative elements composed by intervals' scores be denoted by $N = \langle \text{score}(I_1), \text{score}(I_2), \dots, \text{score}(I_\ell) \rangle$.

Observation 1. Let $a \in I_{k'}$ be the element of indices j in $I^{(k)}$ and j' in $I_{k'}$, $k' \geq k+1$ (an assumption that is tacitly made here is that $I_{k'}$ is a subinterval of $I^{(k)}$). Then,

$$\begin{aligned} \text{score}(I^{(k)}_0^j) &= \text{score}(A_{i_k}^{j_k-1}) + x + a_{j_k} + \text{score}(A_{j_k}^{i_{k'}+j'}) \\ &= x + \text{score}(A_{i_k}^{j_k-1}) + \text{score}((I_{k'})_0^{j'}) \\ &= x + \text{score}(N_{k-1}^{k'-1}) + \text{score}((I_{k'})_0^{j'}) \end{aligned}$$

As an example, take $a = 4$, $I^{(k)} = I^{(1)}$, and $I_{k'} = I_4$ in Figure 5. The equality above indicates the distance of 1 between the curves of $I^{(1)}$ and I_4 for the element $4 \in I_4$.

A first consequence of Observation 1 is a recurrence relation which is used to govern our dynamic programming algorithm. If $k < \ell$, let $I^{(k)} \cap I^{(k+1)}$ stand for the concatenation of the common subintervals of $I^{(k)}$ and $I^{(k+1)}$ (for the sake of illustration, $I^{(1)} \cap I^{(2)} = \langle I_2, I_3, I_4 \rangle$ in the example of Figure 5). In addition, write $I_{k'} \subseteq I^{(k)} \cap I^{(k+1)}$ to say that interval $I_{k'}$ is a common subinterval of $I^{(k)}$ and $I^{(k+1)}$. The recurrence for $\text{score}^*(I^{(k)})$ is given by

$$\text{score}^*(I^{(k)}) = \max\{\text{score}^*(I_k), x + \text{score}(A_{i_k}^{j_k-1})\}, \quad (1)$$

if $k = \ell$ (considering that the last element of A is null) or $(k < \ell$ and $I^{(k)} \cap I^{(k+1)} = \emptyset)$ or, otherwise,

$$\max\{\text{score}^*(I_k), x + \text{score}(A_{i_k}^{j_k-1}), x + \max_{I_{k'} \subseteq I^{(k)} \cap I^{(k+1)}} \{\text{score}(N_{k-1}^{k'-1}) + \text{score}^*(I_{k'})\}\}. \quad (2)$$

The first two terms in (1) and (2) indicate the best insertion index in I_k , while the third one in (2) gives the best interval in $I^{(k)} \cap I^{(k+1)}$ (if any). The crucial point is then the computation of $\max_{I_{k'} \subseteq I^{(k)} \cap I^{(k+1)}} \{score(N_{k-1}^{k'-1}) + score^*(I_{k'})\}$ when I_{k+1} is a subinterval of $I^{(k)}$ (i.e. $I^{(k)} \cap I^{(k+1)} \neq \emptyset$), which is performed in the light of the following additional observations.

Observation 2. Let $a \in I^{(k')}$ be the element of indices j and j' in, respectively, $I^{(k)}$ and $I^{(k')}$, $k' \geq k+1$. Write $I_{k''}$ for the interval containing a , and j'' for the index of a in $I_{k''}$. Assuming that $k'' \neq k'$, then

$$\begin{aligned} score(I^{(k)}_0^j) - score(I^{(k')}_0^{j'}) &= score(N_{k-1}^{k''-1}) + score((I_{k''})_0^{j''}) - score(N_{k'-1}^{k''-1}) - score((I_{k''})_0^{j''}) \\ &= score(N_{k-1}^{k''-1}) \end{aligned}$$

Thus, the respective curves of $I^{(k)}$ and $I^{(k')}$ remain at a constant distance for all intervals $I_{k'} \subseteq I^{(k)} \cap I^{(k+1)}$, $k' \neq k+1$, with the curve of $I^{(k')}$ above that of $I^{(k)}$.

The last observation before going into the details of the algorithm is useful to decide whether a given interval $I_{k'}$ is a subinterval of $I^{(k)}$.

Observation 3. Observation 1 implies that if interval $I_{k'}$, $k' \geq k+1$, is contained in $I^{(k)}$, then $x + score(N_{k-1}^{k'-1}) \geq 0$. The converse is also true since $x + score(N_{k-1}^{k'-1}) \geq 0$ yields $x + score(N_{k-1}^{k''-1}) \geq 0$, for all $k < k'' < k'$, because all members of N are negative.

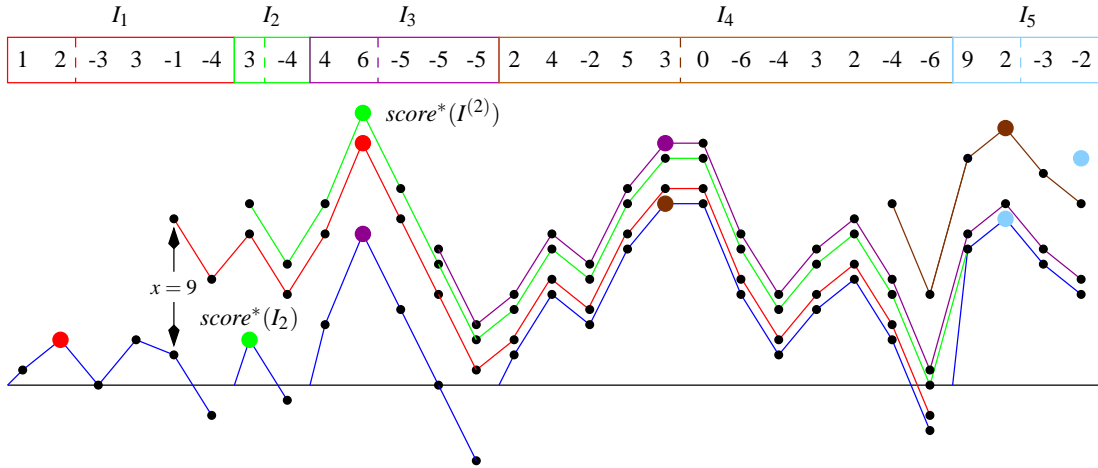


Figure 5: Scores of prefixes of all possible extended intervals resulting from the insertion of $x=9$ in the sequence in Figure 1. For each interval I_k , the points corresponding to $score^*(I_k)$ and $score^*(I^{(k)})$ are highlighted. The last null element of the sequence is omitted.

The computation of the largest scores of prefixes of extended intervals $I^{(k)}$ is divided into two phases. The first phase is a modification of the Kadane's algorithm and its role is twofold. It first determines the largest scores of prefixes of I_1, I_2, \dots, I_ℓ and, then, it sets the initial values of the arrays that are used in the second phase. Such arrays are the following:

SN suffix sums of N , i.e. $SN[k]$ equals $score(N_{k-1}^\ell)$, for all $k \in \{1, 2, \dots, \ell\}$. By definition, $score(N_{k-1}^{k'-1}) = SN[k] - SN[k']$, for all $k' \geq k$.

INTSCR largest intervals' scores, i.e. $INTSCR[k] = score^*(I_k)$, for all $k \in \{1, 2, \dots, \ell\}$.

XSCR for each interval $k \in \{1, 2, \dots, \ell\}$, this array stores the score of the subsequence ending at x , provided that x is inserted in I_k , i.e. $XSCR[k] = x + score(A_{I_k}^{j_k-1})$.

The second phase is devoted to the computation of the extended interval containing the best insertion position for x . This is done iteratively from $k=1$ until $k=\ell$. For each k , the recurrence relation (1)–(2)

is used to start the computation of $\text{score}^*(I^{(k)})$ and to update the maximum score of extended intervals started in previous iterations as described in Algorithm 2. Such information is stored as follows. The array EXTSCR contains the maximum scores of prefixes of the extended intervals $I^{(k')}$, for all $k' \in \{1, 2, \dots, k\}$. The intervals with best prefix scores obtained so far are kept in the queue INTQ . Q is the rear of the queue INTQ , initialized at 0.

Algorithm 2: Second phase for the case $x > 0$

Input: Arrays SN , INTSCR , and XSCR computed in the first phase

Output: An optimal insertion interval for A

```

1  $k \leftarrow 1$ 
2  $\text{EXTSCR}[k] \leftarrow \max\{\text{INTSCR}[k], \text{XSCR}[k]\}$ 
3  $Q \leftarrow 1$ 
4  $\text{INTQ}[Q] \leftarrow k$ 
5 for  $k \leftarrow 2, \dots, \ell$  do
6    $\text{DIST} \leftarrow x + \text{SN}[\text{INTQ}[Q]] - \text{SN}[k]$ 
7   while  $\text{DIST} \geq 0$  and  $\text{DIST} + \text{INTSCR}[k] > \text{EXTSCR}[\text{INTQ}[Q]]$  do
8      $\text{EXTSCR}[\text{INTQ}[Q]] \leftarrow \text{DIST} + \text{INTSCR}[k]$ 
9     if  $Q > 1$  and  $\text{EXTSCR}[\text{INTQ}[Q]] \geq \text{EXTSCR}[\text{INTQ}[Q-1]]$  then
10       $Q \leftarrow Q - 1$ 
11       $\text{DIST} \leftarrow x + \text{SN}[\text{INTQ}[Q]] - \text{SN}[k]$ 
12    $\text{EXTSCR}[k] \leftarrow \max\{\text{INTSCR}[k], \text{XSCR}[k]\}$ 
13   if  $\text{EXTSCR}[k] < \text{EXTSCR}[\text{INTQ}[Q]]$  then
14      $Q \leftarrow Q + 1$ 
15      $\text{INTQ}[Q] \leftarrow k$ 
16 return  $\text{INTQ}[Q]$ 

```

The correctness of the two-phase algorithm stems from the following lemma.

Lemma 2. For every iteration k (just before execution of line 5 of Algorithm 2), let I_k be an interval and $k'' = \text{INTQ}[Q]$. Then, the following conditions hold:

1. $\text{EXTSCR}[k''] = \text{score}^*(I^{(k'')} \setminus A_{j_k}^{j_\ell})$;
2. if $Q > 1$ and k' appears in INTQ but $k'' \neq k'$, then $k' < k''$ and $\text{score}^*(I^{(k'')} \setminus A_{j_k}^{j_\ell}) < \text{score}^*(I^{(k')} \setminus A_{j_k}^{j_\ell})$; and
3. if $k' < k$ does not appear in INTQ , then k'' is such that $\text{score}^*(I^{(k'')} \setminus A_{j_k}^{j_\ell}) \leq \text{score}^*(I^{(k')} \setminus A_{j_k}^{j_\ell})$.

Proof. By induction on k . For $k = 1$, condition 1 holds trivially due to line 2, while conditions 2 and 3 hold by vacuity. Let $k > 1$. We need to analyze the changes in INTQ . We start with the intervals that are removed from INTQ . At line 6, Observation 1 is used to compute the distance between the curves of $I^{(k'')}$ and I_k . If this distance is negative, then I_k is not a subinterval of $I^{(k'')}$. Otherwise, condition 1 of the induction hypothesis is used in the comparison of line 7 and $\text{EXTSCR}[k'']$ is updated at line 8 according to (2) using Observation 1. So, condition 1 remains valid for k up to this point of the execution. If $\text{EXTSCR}[k'']$ increases (i.e. line 8 is executed), then Observation 2 and condition 2 of the induction hypothesis are evoked to remove $I^{(k'')}$ from the queue respecting condition 3 in case a point of I_k in the curve of $I^{(k'')}$ overcomes that of an interval that precedes $I_{k''}$. $\text{EXTSCR}[k'']$ is updated again according to (2) in order to satisfy condition 1. This procedure is repeated until condition 2 is valid for the intervals still in INTQ .

Finally, lines 12–15 correspond to the insertion in INTQ . The maximum score of the prefix of $I^{(k)}$ containing I_k and x only is updated at line 12 and $I^{(k)}$ enters the queue only if such maximum score is

below the maximum score of the prefix of $I^{(INTQ[Q])}$ considered so far. This implies that conditions 2 and 3 are also valid for k . \square

Theorem 1. *The ISS problem can be solved in $O(n)$ time and space.*

5 Sorting

We now turn our attention to the SSS problem. Its hardness is analyzed considering the following derived problem.

Restricted version of the SSS problem: given two positive integers k and s , we denote by $SSS(k, s)$ the restricted version of the SSS problem where $n = 3k$, the elements in A are integers and bounded by a polynomial function of k , all negative elements are equal to $-s$, and every positive element a_i is such that $s/4 < a_i < s/2$.

A consequence of the fact that sorting a sequence is similar to accommodate the positive elements in order to create an appropriate partition into intervals leads to the following result.

Theorem 2. *The $SSS(k, s)$ problem is strongly NP-hard.*

Proof. By reduction from the 3-PARTITION decision problem, stated as follows: given $3k$ positive integers a_1, \dots, a_{3k} , all polynomially bounded in k , and a threshold s such that $s/4 < a_i < s/2$ and $\sum_{i=1}^{3k} a_i = ks$, there exist k disjoint triples of a_1 to a_{3k} such that each triple sums up to exactly s ? 3-PARTITION problem is known to be NP-complete in the strong sense [12].

Given an instance C of the 3-PARTITION problem, an instance of the $SSS(k, s)$ problem is defined by an arbitrary permutation A of the multiset C' obtained from C by the inclusion of k occurrences of $-s$. A solution for the SSS instance is to choose elements of C for each negative element of C' , which gives a partition of C . Since $a_i > s/4$, for all $i \in \{1, \dots, 3k\}$, every sequence of 4 positive elements chosen from C' has value greater than s . Thus, C is a “yes” instance of the 3-PARTITION problem if and only if $score^*(A') = s$. \square

We show in the sequel that Algorithm 3 is a parametrized approximation algorithm for the SSS problem. Such an algorithm builds a permutation of A keeping the maximum scoring subsequence of all intervals, except the last one, bounded by the input parameter plus the largest element of A . For the last interval, the following holds for every sequence A .

Observation 4. *If $N = \langle score(I_1), score(I_2), \dots, score(I_\ell) \rangle$ is the sequence of negative elements composed by intervals' scores, then $score(N_{\ell-1}^\ell) = score(A) - score(N_0^{\ell-1})$. Considering that I_ℓ is a subsequence of A and that $score(N_0^{\ell-1}) < 0$, we conclude that $score(N_{\ell-1}^\ell)$ is a lower bound for $score^*(A)$ at least as good as $score(A)$.*

Algorithm 3 gets as input, in addition to the instance A (with size n), the parameter L , which depends on $M = \max_{a \in A} a$. A variable S is used to keep the score of the interval being currently constructed. Just after step 10 is executed, it turns out that $L + M \geq S \geq L$. On the other hand, execution of step 15 leads to $S < L$ or includes all remaining negative elements in A' . After that, if $S + score(Q) + score(R) < 0$, then a new interval I_k is established and S is incremented by $score(N_{k-1}^k)$. A straightforward consequence is that $score^*(A') > L + M$ only if step 16 is executed with positive elements of A , and this due to the last interval (in the sense of Observation 4). This leads to the following result.

Lemma 3. *Let A be an instance of the SSS problem, A' be the sequence returned by the call $PARAMETRIZED_SORTING(A, L)$, for some $L \geq M$, and N' be the sequence of the scores of the ℓ' intervals of A' . Then,*

$$score^*(A') \leq \max\{0, L + M, score(N_{\ell'-1}^{\ell'}) = score(A) - score(N_0^{\ell'-1})\}. \quad (3)$$

Moreover, $PARAMETRIZED_SORTING(A, L)$ runs in $O(n)$ time.

Algorithm 3: PARAMETRIZEDSORTING(A, L)

Input: an array A of $n \geq 0$ numbers and a parameter $L \geq M$

Output: an array A' containing a permutation of A

```
1 Let  $A'$  be an array of size  $n$  and
2 Let  $A^- \subseteq A$  and  $A^+ \subseteq A$  be sequences of negative and positive members of  $A$ , respectively
3  $j \leftarrow 1$ 
4  $S \leftarrow 0$ 
5 while  $A^- \neq \emptyset$  and  $A^+ \neq \emptyset$  do
6   Let  $Q$  be a sequence of elements of  $A^+$  such that  $L \leq S + \text{score}(Q) \leq L + M$ , if one exists, or
    $Q = A^+$  otherwise
7   Assign the elements of  $Q$  to  $A'[j \dots j + |Q| - 1]$ 
8    $j \leftarrow j + |Q|$ 
9
10   $A^+ \leftarrow A^+ \setminus Q$ 
11  Let  $R$  be a minimal sequence of elements of  $A^-$  such that  $S + \text{score}(Q) + \text{score}(R) < L$ , if one
   exists, or  $R = A^-$  otherwise
12   $S \leftarrow \max\{0, S + \text{score}(Q) + \text{score}(R)\}$ 
13  Assign the elements of  $R$  to  $A'[j \dots j + |R| - 1]$ 
14   $j \leftarrow j + |R|$ 
15   $A^- \leftarrow A^- \setminus R$ 
16 Assign the elements of  $A^- \cup A^+$  to  $A'[j \dots |A^- \cup A^+| - 1]$ 
17 return  $A'$ 
```

The key of our approximation algorithm is to provide Algorithm PARAMETRIZEDSORTING with an appropriate lower bound parameter. The most immediate one is $L = \max\{0, M, \text{score}(A) - M\}$, which, however, does not capture the contribution of the negative members of A whose values are smaller than $-L$ when A contains at least one nonnegative element. In order to circumvent this difficult case of Lemma 3, assume that A^* is an optimum solution and $OPT = \text{score}^*(A^*)$. According to (3), we need to find a new value for L such that $\text{score}(N_{\ell'-1}^{\ell'}) \leq L \leq OPT$, being $I_{\ell'}$ the last interval of the sequence A' returned by PARAMETRIZEDSORTING(A, L), with the purpose of having $\text{score}^*(A') \leq 2OPT$.

We first argue that we can assume that the last element a^* of A^* is such that $a^* \geq -OPT$: if $M \geq 0$ and $a^* < -OPT$, then construct optimum solution from A^* by moving a^* to the first position. Since the first interval of the new sequence contains a^* only, the scores of the remaining subsequences are not affected. Hence, the new sequence is still optimum. If necessary, repeat this operation until the last element is at least as large as $-OPT$. The optimum sequence so obtained is such that if $a < -L$, then a is not in the last interval. It follows that either $\max\{0, M, \text{score}(A) - M\} = 0$ or, by Observation 4, a strengthened lower bound L can be chosen satisfying the inequality

$$\begin{aligned} L &\geq \text{score}(A) + \sum_{a_i \in B_L} (-a_i - L) \\ &= \text{score}(A \setminus B_L) - L|B_L|, \end{aligned}$$

where $B_L = \{a_i \in A \mid a_i < -L\}$.

Writing the inequality for L as

$$L \geq \frac{\text{score}(A \setminus B_L)}{|B_L| + 1}, \quad (4)$$

we define the two-phase Algorithm APPROXSORTING(A). Its first phase consists in determining the smallest L satisfying (4) and the conditions of Lemma 3. To do so, set $L = \max\{0, M, \text{score}(A) - M\}$ and take the elements of a decreasing sequence P on the set $\{-a \mid a \in A, a < -L\}$ (note that, by definition, all elements of P are distinct). If $L > 0$ and $P \neq \langle \rangle$, then write this sequence as $P = \langle p_1, p_2, \dots, p_{|P|} \rangle$, find

the maximal index k (in the range from 1 to $|P|$) such that $p_k \geq \text{score}(A \setminus B_{p_k})/k$, and set $L = \text{score}(A \setminus B_{p_k})/k$. The second phase is simply a call $\text{PARAMETRIZED_SORTING}(A, L)$ to produce a permutation A' of A . Since each interval I'_k of A' having $\text{score}(I'_k) < 0$ has an $a \in B_L$ as last element, we get $\text{score}(I'_k) \geq a + L$. Observation 4 leads to $\text{score}(N_{\ell'-1}^{\ell'}) \leq L$. Therefore, Lemma 3 gives the following result.

Theorem 3. *APPROXSORTING is a 2-approximation algorithm for the SSS problem and a $(3k+1)/2k$ -approximation algorithm for the $\text{SSS}(k, s)$ problem which runs in $O(n \log n)$ time.*

Proof. The approximation factors stem directly from Lemma 3 and $M \leq \text{score}^*(A)$. In special, for the $\text{SSS}(k, s)$ case, inequality 4 gives $L = ks/(k+1)$. Considering that $s/2$ is an upper bound for all positive elements, we get $1 + M/\text{score}^*(A) < (3k+1)/2k$, which leads to the claimed approximation factor for this case. \square

A final remark that can be made in connection with algorithm APPROXSORTING for the SSS problem is that the approximation factor of 2 is tight. To see this, consider $x > 0$ and $x/2 < y < x$. The sequence A returned by the call $\text{PARAMETRIZED_SORTING}(\langle y, -x, y, -x, x \rangle, x)$ is either $\langle y, y, -x, x, -x \rangle$, or $\langle y, x, -x, y, -x \rangle$, or $\langle x, -x, y, y, -x \rangle$. It follows that $2y \leq \text{score}^*(A) \leq y + x$. Then, since $\text{OPT} = x$, $\frac{\text{score}^*(A)}{\text{OPT}} \rightarrow 2$ as $x - y \rightarrow 0$.

6 Concluding remarks

We investigate two problems related to maximum scoring subsequences of a sequence, namely the INSERTION IN A SEQUENCE WITH SCORES (ISS) and SORTING A SEQUENCE BY SCORES (SSS) problems. For the ISS problem, we provided a linear time solution, and for the SSS one we proved its NP-hardness and give a 2-approximation algorithm. An additional remark with regard to the ISS problem is that our solution extends immediately to its *circular* version, where to the set of subsequences we add each $\langle A_i^n, A_0^j \rangle$ such that $1 \leq j \leq i < n$. The core of this extension is to modify Kadane's algorithm so that it takes the new subsequences into account as well. This can be done by means of a preliminary $O(n)$ step computing $E[i] = \max_{i \leq k < n} \{\text{score}(A_k^n)\}$ for each $i \in \{1, \dots, n-1\}$, so that $E[j] + \text{score}(A_0^j)$ gives the maximum sum of a *circular* subsequence ending in a_j , for each $j \in \{1, \dots, n-1\}$. We can then solve the circular version of the ISS problem as follows: if $x < 0$, we first find a (circular or not) maximum scoring subsequence S of A which is minimal in size, and then proceed as before to find an optimal insertion position inside of S . If $x > 0$, then we first use the extended version of Kadane's algorithm to find the partition of A into *possibly circular* intervals – the interval containing a_n continues with elements a_1, a_2, \dots, a_k for some $k \in \{0, \dots, n-1\}$; next, we build a circularly shifted permutation A' of A by moving elements a_1, a_2, \dots, a_k to the end of the sequence, and then apply the algorithm for the non-circular case of ISS to A' , which clearly also gives a solution for A .

The SSS problem is also closely related to another set partitioning problem, called MULTIPROCESSOR SCHEDULING problem, stated as follows: given a multiset C of positive integers and a positive integer m , find a partition of C into m subsets C_0, C_1, \dots, C_{m-1} such that $\max_{i \in \{0, 1, \dots, m-1\}} \{\sum_{a \in C_i} a\}$ is minimized. Not surprisingly, given an instance (C, m) of MULTIPROCESSOR SCHEDULING, an instance of the SSS problem can be defined as an arbitrary permutation A of the multiset C' obtained from C by the inclusion of $m-1$ occurrences of the negative integer $-\text{score}(C) - 1$, indicating that a solution of the SSS problem for A induces a solution of the MULTIPROCESSOR SCHEDULING problem for (C, m) . This problem admits a polynomial time approximation scheme (PTAS) [13, 14] as well as list scheduling heuristics producing a solution which is within a factor of $2 - 1/n$ (being n the number of elements in the input multiset C) from the optimal [15]. On the other hand, MAX-3-PARTITION, the optimization version of the problem used in the proof of Theorem 2, is known to be in APX-hard [16]. A natural open question is, thus, whether there exist a polynomial time approximation algorithm with factor smaller than 2 for the SSS problem. In this regard, note that although transferring our approximation factor from the SSS problem to the MULTIPROCESSOR SCHEDULING problem is easy, the converse appears

harder to be done, since we do not know in advance how many intervals there should be in an optimal permutation A' of A .

The SSS problem has the following generalization for multidimensional vectors of numbers, which arises in the context of buffer minimization in radio networks. Given a matrix M , let $row_M(i)$ and $col_M(j)$ denote respectively the i -th row and the j -th column of M . The multidimensional version of the SSS problem is then that of, given a $k \times n$ matrix M , with $k, n > 0$, finding a permutation M' of the columns of M which minimizes $value(M') = \sum_{i=1}^k score^*(row_{M'}(i))$. This problem is hard even for very restricted matrices.

Theorem 4. *The multidimensional version of the SSS problem is NP-hard even if $M[i, j] \in \{-1, +1\}$ for all i and j .*

Proof. Consider the following polynomial-time reduction from the HAMILTONIAN PATH problem, which is NP-complete [12]. Given an undirected graph $G = (V, E)$, with $V = \{1, \dots, n\}$, let $\bar{E} = \{S \subseteq V : |S| = 2 \text{ and } S \notin E\} = \{\{x_1, y_1\}, \dots, \{x_k, y_k\}\}$. We assume that $k \geq 1$ because otherwise the HAMILTONIAN PATH problem is trivial. Define M as the $k \times n$ matrix such that $M[i, j] = +1$ if $j \in \{x_i, y_i\}$ and $M[i, j] = -1$ otherwise. If there is a Hamiltonian path $P = \langle \ell_1, \dots, \ell_n \rangle$ in G , then let M' be the permutation of the columns of M according to P . It turns out that $score^*(row_{M'}(i)) \geq 1$ for all i , since $M'[i, j] = +1$ for each $\ell_j \in \{x_i, y_i\}$, which means that $value(M') \geq k$. However, it cannot be the case that $score^*(row_{M'}(i)) > 1$ for some i , since otherwise there is j such that $M'[i, j] = M'[i, j+1] = +1$, which implies that $\{\ell_j, \ell_{j+1}\} \in \bar{E}$ and thus contradicts our assumption about P . Therefore $value(M') = k$. Conversely, if G has no Hamiltonian path, then, as in the previous case, $score^*(row_{M'}(i)) \geq 1$ for all i . However, since any permutation $P = \langle \ell_1, \dots, \ell_n \rangle$ is not a Hamiltonian path of G , then there are j and i such that $\{\ell_j, \ell_{j+1}\} = \{x_i, y_i\} \in \bar{E}$, which implies that $score^*(row_{M'}(i)) \geq 2$. Therefore $value(M') \geq (k-1) + 2 > k$, completing the proof. \square

Note, however, that the same is not true for the particular case $k = 1$ of the multidimensional SSS problem: the SSS problem is polynomially solvable to optimality when every element $a_i \in A$ is such that $a_i \in \{\beta, \pi\}$, where β, π are two real numbers [11]. We leave it as an open question whether or not there are constant factor approximation algorithms for the multidimensional version of the SSS problem in its general form. A final remark is that the ISS problem can be generalized in a similar way for the case of vectors of numbers: given a $k \times n$ matrix M and a $k \times 1$ column X , find a position p for inserting X into M that minimizes the value of the resulting matrix. Like the ISS problem, this one admits a trivial polynomial solution: checking all the $n+1$ possibilities and choosing an optimal one takes $O(n^2 \cdot k)$. We wonder, however, whether more efficient algorithms exist.

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